Problem Set 2
Algorithms Advanced Course TDA250

Per Lindstrand

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Introduction

Problem Set 2 by Per Lindstrand (`muusu@dtek.chalmers.se`), 820921-5576.

[R] Problem 2.1 (Auctions) [KT 13.9]

Given $n = 2m$ random bids $b_1, \ldots, b_n$ we want to accept the highest bid b^* with probability at least 1/4. Remember the highest bid b' in the first half of the bids, that is $b' = \max\{b_1, \ldots, b_m\}$. Accept the first bid $b_i \in \{b_{m+1}, \ldots, b_n\}$ such that $b_i > b'$.

Proposition 1.0.1 *The algorithm accepts the highest bid with probability at least* 1/4.

Proof Consider the following events

(a) The highest bid b^* is in the first half.

(b) The highest bid b^* is in the second half.

(c) The next to highest bid b^{*-} is in the first half.

(d) The next to highest bid b^{*-} is in the second half.

As the bids are made in random order the probability that any bid b_i is in either half of the bids is 1/2. Hence, all of the above events have equal probability 1/2. Events (a) and (b) are mutually exclusive, as are (c) and (d). Therefore, there are four possible events

(A) b^* and b^{*-} are in the first half.

(B) b^* and b^{*-} are in the second half.

(C) b^* is in the first half and b^{*-} is in the second half.

(D) b^* is in the second half and b^{*-} is in the first half.

We have that $P[A] = P[a]P[c]$, $P[B] = P[b]P[d]$, $P[C] = P[a]P[d]$ and $P[C] = P[b]P[c]$. Since $P[a] = P[b] = P[c] = P[d] = 1/2$ we have that $P[A] = P[B] = P[C] = P[D] = 1/4$. The algorithm will always accept the highest bid if (D) occurs. The algorithm will also accept the highest bid in certain instances of event (B). If, for instance, the third highest bid is in the first half and the next to highest is *after* the highest in the second half and so on. Therefore, the algorithm accepts the highest bid b^* with probability at least 1/4. ∎

[R] Problem 2.2 MAX-CUT

Proposition 1.0.2 *The optimal maximum cut E^* of a graph $G = (V, E)$ is bound by $|E^*| \leq |E|$.*

Proof As an edge can only be cut once the maximum cut can never exceed the number of edges in the graph. Therefore, $|E^*| \leq |E|$. ∎

Proof The algorithm colours each vertex $v \in V$ with a colour $C(v)$, either red or blue with equal probability 1/2. This yields a binary partition of the graph. Consider an edge $e \in E$, let $X_e = 1$ if $C(u) \neq C(v)$ where $u, v \in e$ and $X_e = 0$ otherwise. We have that

$$\begin{aligned} P[X_e = 0] &= P[C(u) = C(v)] \\ &= \underbrace{P[C(u) = r]\,P[C(v) = r]}_{1/4} + \underbrace{P[C(u) = b]\,P[C(v) = b]}_{1/4} \\ &= 1/2 \end{aligned}$$

and

$$\begin{aligned} P[X_e = 1] &= P[C(u) \neq C(v)] \\ &= \underbrace{P[C(u) = r]\,P[C(v) = b]}_{1/4} + \underbrace{P[C(u) = b]\,P[C(v) = r]}_{1/4} \\ &= 1/2 \end{aligned}$$

which yields

$$\begin{aligned} E[X_e] &= \sum_x x P[X_e = x] \\ &= 0 \cdot P[X_e = 0] + 1 \cdot P[X_e = 1] \\ &= P[X_e = 1] \\ &= 1/2 \end{aligned}$$

Consider all edges in the graph and the expected size of the cut given by the algorithm

$$E\left[\sum_{e \in E} X_e\right] = \sum_{e \in E} E[X_e] = \sum_{e \in E} \frac{1}{2} = \frac{1}{2}|E|$$

The expected cut size will be half of the edges in the graph. Let the E' denote the solution that the algorithm produces, then

$$|E'| \geq \frac{1}{2}|E^*|$$

where $|E'| = \frac{1}{2}|E|$

$$\frac{1}{2}|E| \geq \frac{1}{2}|E^*|$$

which is exactly the optimal bound $|E^*| \leq |E|$. ∎

[MC] Problem 2.3

Part a

Let $S = \{1, \ldots, 6\}$ be the state space and

$$P = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

be the transition matrix.

Part b

Let $S = \mathbb{N}$ be the state space and

$$P(i \to j) = \begin{cases} 1/6 & \text{if } i < j \leq i + 6 \\ 0 & \text{otherwise} \end{cases}$$

be the transition mechanism.

To clarify, for any state X_n we have that $X_{n+1} = X_n + Y$ where Y is a random variable taking values $\{1, 2, 3, 4, 5, 6\}$ each with probability 1/6. That is, by rolling the die one more time it is possible to move to states within $X_n + 1, \dots, X_n + 6$, each with probability 1/6.

Part c

Let $S = \mathbb{N}$ be the state space and

$$P(i \to j) = \begin{cases} 5/6 & \text{if } j = i + 1 \\ 1/6 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

be the transition mechanism.

For any state we have a probability of 5/6 to not roll a 6 and therefore increment the time since the last 6. Corollary, we have a probability of 1/6 to roll a 6 and therefore to return to state 0.

Part d

Let $S = \mathbb{N}$ be the state space and

$$P(i \to j) = \begin{cases} (5/6)^j(1/6) & \text{if } i = 0 \\ 1 & \text{if } i > 0 \text{ and } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

or perhaps

$$P(i \to j) = (5/6)^j(1/6)$$

be the transition mechanism.

The former is a more interesting interpretation, if we are in state i we have i rolls until the next 6 therefore the next state must be $i - 1$. In state 0 we have a probabilistic time until the next 6.

[MCMC] Problem 2.4 [KT 8.9.6]

Part a

Let v be any fixed vertex in V. Let ξ and ζ be two states such that

$$\xi(w) = \zeta(w) \qquad \text{for all } w \in V - \{v\}$$

then

$$P(\xi \to \zeta) = \begin{cases} \frac{1}{|V|}\left(1 - \frac{1+\sum_{q\in Q} I_{\{\xi(u)|(u,v)\in E\}}(q)}{|Q|}\right) & \text{if } \xi(v) \neq \zeta(v) \\ \sum_{v\in V} \frac{1}{|V|}\left(\frac{1+\sum_{q\in Q} I_{\{\xi(u)|(u,v)\in E\}}(q)}{|Q|}\right) & \text{if } \xi(v) = \zeta(v) \end{cases}$$

where

$$I_S(s) = \begin{cases} 1 & \text{if } s \in S \\ 0 & \text{otherwise} \end{cases}$$

and for any two states ξ and η such that $\xi(v) \neq \eta(v)$ for at least two $v \in V$ we have

$$P(\xi \to \eta) = 0$$

since we do not allow two nodes to change colour at the same time.

Part b

Given a colouring ξ and a node $v \in V$ it is always possible to select a colour $c \in Q$ such that $\xi(v) = c$. This event is a self-loop in the Markov chain. Furthermore there are certain *illegal* colourings which results in self-loops.

Part c

Definition Let $|v|$ denote the *degree* of v, the number of incident edges to v.

Definition Let $C(v)$ denote the *colour* of v.

Proposition 1.0.3 *Given $q \geq \Delta + 2$ it is always possible to change the colour of any vertex $v \in V$.*

Proof For every edge $(u, v) \in E$ vertices u and v may not have the same colour. For all incident edges (u, v) to v we have that for any colouring ξ $\xi(v) \neq \xi(u)$ therefore, in order to colour v and all neighbours to v we need at most $|v| + 1$ colours.

Given $|v| + 2$ colours there is always at least one unused colour, not used by v or any neighbour of v. This colour, say c', can be used to "transfer" the colour c_u of any neighbour u of v to v.

1. Let $C(u) = c'$
2. Let $C(v) = c_u$
3. (Optional.) Let $C(u) = c_v$

Since the maximum degree in the graph G is Δ we know that we can "transfer" any colour $c \in Q$ to any vertex $v \in V$ given $q = \Delta + 2$ colours. ∎

Proposition 1.0.4 *Given $q \geq \Delta + 2$ the Markov chain is irreducible.*

Proof For any two states $\xi, \zeta \in S$ there is a path from ξ to ζ since we can change the colour of any vertex $v \in V$ to any colour $c \in Q$ according to 1.0.3. Hence, the Markov chain is irreducible. ∎

Part d

Proposition 1.0.5 *For any states* $\xi, \zeta \in S$, $P_{\xi,\zeta} = P_{\zeta,\xi}$.

Proof According to the transition mechanism if $\exists u, v \in V$ such that $\xi(v) \neq \zeta(v)$ and $\xi(u) \neq \zeta(u)$ then $P_{\xi,\zeta} = P_{\zeta,\xi} = 0$. Further, if $\xi(v) = \zeta(v)$ for all $v \in V$ then of course $P_{\xi,\zeta} = P_{\zeta,\xi}$ since $P_{\eta,\eta} = P_{\eta,\eta}$.

Let $v \in V$ be any fixed vertex such that $\xi(v) \neq \zeta(v)$ and let $\xi(w) = \zeta(w)$ for all $w \in V - \{v\}$. Let Q_v denote the set of *legal* colours for v. Clearly $\xi(v) \in Q_v$ and $\zeta(v) \in Q_v$ (since both are valid colourings.) In both states one colour from Q_v is used to colour v. Therefore

$$|Q_v - \{\xi(v)\}| = |Q_v - \{\zeta(v)\}|$$

which means that the number of possible colours remains the same. The transition probability

$$\frac{1}{|V|}\left(1 - \frac{1 + \sum_{q \in Q} I_{\{\xi(u)|(u,v)\in E\}}(q)}{|Q|}\right) = \frac{1}{|V|}\left(1 - \frac{1 + \sum_{q \in Q} I_{\{\zeta(u)|(u,v)\in E\}}(q)}{|Q|}\right)$$

since $\{\xi(u)|(u,v) \in E\} = \{\zeta(u)|(u,v) \in E\}$. Hence, $P_{\xi,\zeta} = P_{\zeta,\xi}$. ∎

Since the Markov chain is *aperiodic* and *irreducible* there is a unique stationary distribution. Assume a uniform distribution $\pi_\xi = \frac{1}{|S|}$ for all $\xi \in S$.

Proposition 1.0.6 π *is reversible.*

Proof If π is a reversible distribution on S then

$$\pi_\xi P_{\xi,\zeta} = \pi_\zeta P_{\zeta,\xi}$$

for all $\xi, \zeta \in S$. We know that $P_{\xi,\zeta} = P_{\zeta,\xi}$ and $\pi_\xi = \pi_\zeta$ for all $\xi, \zeta \in S$ since π is a uniform distribution. ∎

Part e

We use a Monte Carlo sampling algorithm in order to solve the q-colouring problem of a graph $G = (V, E)$. Let $S_0, \ldots, S_n$ be subsets of the solution space such that $S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_0$ where $S_0 = U$ and $S_n = S$. Each subset consists of all solutions to the q-colouring problem for a modified graph G' with a subset of the edges $E' \subset E$. By adding one edge for each subset starting with no edges when $S_0 = U$. We have that

$$|S| = |S_n| = \frac{|S_n|}{|S_{n-1}|}\frac{|S_{n-1}|}{|S_{n-2}|}\cdots\frac{|S_1|}{|S_0|}|S_0|$$

where $S_0 = q^{|V|}$. Let $\rho_i = \frac{|S_i|}{|S_{i-1}|}$ then

$$|S| = \rho_n \rho_{n-1} \cdots \rho_1 q^{|V|}$$

We want to approximate ρ_n with $\tilde{\rho}_n$ we can estimate $|S|$. To estimate ρ_i we apply Monte Carlo sampling on S_{i-1}. We generate points in S_{i-1} using a Markov chain with state space $\Omega_{i-1} = S_{i-1}$ and the transition mechanism described earlier but only allow transitions $\xi \to \zeta$ if $\zeta \in S_{i-1}$.

This Markov chain is *aperiodic*, *irreducible*, *reversible* and has a unique stationary distribution $\pi^* = \left\{ \frac{1}{|S_{i-1}|}, \ldots, \frac{1}{|S_{i-1}|} \right\}$. By running the Markov chain for a large number of steps T

$$\pi^T \approx \pi^*$$

regardless from which state we start.

Algorithm 1 MCMC-q-Colouring

for $i = 1$ to $n - 1$ **do**
    $s = 0$
    for $r = 1$ to k **do**
        Let M_{i-1} be a Markov chain with state space $\Omega_{i-1} = S_{i-1}$ and transition mechanism P
        Run Markov chain M_{i-1} for T steps to get point ω
        if $\omega \in S_i$ **then**
            $s = s + 1$
        end if
    end for
    Let $\tilde{\rho}_i = s/k$
end for
return $\tilde{\rho}_1 \tilde{\rho}_2 \cdots \tilde{\rho}_{n-1} q^{|V|}$