Problem Set 3
Algorithms Advanced Course TDA250

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Introduction

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[NP] Problem 3.1 (NP-completeness) [KT 8.20]

Proposition 1.0.1 (Low-Diameter-Clustering $\in \mathcal{NP}$) *There exists a polynomial time certifier for a solution S to an instance $(\{p_1, \ldots, p_n\}, B, d(\cdot,\cdot), k)$ of* Low-Diameter-Clustering*.*

Proof Given a solution S to Low-Diameter-Clustering consisting of m subsets $P_1, \ldots, P_m$ it is possible to verify that S is correct in polynomial time. Verify that $m \leq k$ and then ensure that $d(p_i, p_j) \leq B$ for all $p_i, p_j \in P_l$ for $1 \leq l \leq m$. This is indeed polynomial. In the worst case we have to compare all objects to all other objects, that is $n(n-1)/2$ comparisons if they're all in the same subset. This is $O(n^2)$. ∎

Proposition 1.0.2 (Low-Diameter-Clustering $\mathcal{NP}$**-complete**) *Given a set of n objects $\{p_1, \ldots, p_n\}$, a function $d(\cdot,\cdot)$, a distance B and a number k, the problem of* Low-Diameter-Clustering *is $\mathcal{NP}$-complete.*

Consider the reduction k-Colour $\leq_{\mathcal{P}}$ Low-Diameter-Clustering. Given a black box to solve Low-Diameter-Clustering and an instance of k-COLOUR with a graph $G = (V, E)$ and a number k, is there a colouring ν of all vertices using at most k colours such that $\nu(u) \neq \nu(v)$ if $(u, v) \in E$?

Let all vertices $v \in V$ correspond to objects p_v. Define the function $d(\cdot,\cdot)$ such that

$$d(p_v, p_u) = \begin{cases} cB & \text{if } (u,v) \in E \\ B/c & \text{if } (u,v) \notin E \\ 0 & \text{if } u = v \end{cases}$$

with some constant $c > 1$ for any $B > 0$. This concludes the construction.

Proof ($\rightarrow$) If there is a solution S to the Low-Diameter-Clustering problem then it is also a solution to k-Colour. Since no subset $P \in S$ contains any objects $p_v, p_u \in P$ such that $d(p_v, p_u) > B$ we know that $(u, v) \notin E$. Therefore, u and v can have the same colour. Colour the vertices in such a way that for $p_v \in P_i$ let $\nu(v) = C(P_i)$ where $C(\cdot)$ is some unique colour for each subset $P_i \in S$ such that $C(P_i) \neq C(P_j)$ if $i \neq j$. ∎

Proof ($\leftarrow$) If there is a solution to the k-Colour problem then it is also a solution to Low-Diameter-Clustering. Since $\nu(v) \neq \nu(u)$ for any vertices $u, v \in V$ such that $(u, v) \in E$ we know that $d(p_v, p_u) > B$. Therefore, p_v and p_u can not be in the same subset $P \in S$. Cluster the objects such that for each colour $c_1, \ldots, c_k$ define subsets $P_1, \ldots, P_k$ and let $p_v \in P_i$ if $\nu(v) = c_i$. ∎